be the order that minimizes

$$\left\{\sum_{k=115}^{125}|u_i(k)|,\ \forall i\right\}$$

where $u_i(k)$ is the control value obtained if the selected plant order were i. Notice that this period is the last 10 steps of the learning period, during which a proportional controller is used in real time. Once an order is chosen, the adaptive controller can be started. The motivation is to choose the order that results in the smallest amount of energy added to the system. The order that minimized

$$\left\{\sum_{k=115}^{125}|u_i(k)|\right\}$$

was 5. Figure 4 shows the system response when the order was set to 5 after the impact, but this order was allowed to vary (a fixed order at 5 would result in a plot similar to Fig. 2). The estimated value for the order is increased automatically from 5 to 7 and stays at 7 for approximately 50 steps, before returning to 6. The response obtained by this approach is at least as good as the response in Fig. 3, where the initial order of the controller was set to 7. Clearly, this provides only a partial answer to the previous question, in the sense that this approach was motivated, and works best, for this particular control law. We anticipate that more refined methods will result from further research. Nonetheless, it points to the importance of the choice of the control law in determination of the appropriate order for the adaptive controller.

A few issues concerning the implementation are noteworthy. Throughout the simulation, the lattice filter generates parameter estimates for all plant orders $N \leq N_{max} = 8$. The lattice is not reinitialized after the impact, but the forgetting factor 0.99 is used to discard old data exponentially. The impact itself is detected by monitoring the position of the second mass. In all of the plots shown, we have used $u_{sat} = 35$ and $u_{max} = 100$. We have found that changing the value of the u_{sat} can result in different system behavior (e.g., the plots in Figs. 2–4 would be substantially different from $u_{sat} = 25$). The performance is much less sensitive to changes in u_{max}, however. In this example, u_{max} can vary from 70 to 130, whereas Fig. 2 stays exactly as shown here. For simplicity, real values were chosen for the desired poles (i.e., roots, of A^*). Complex poles, placed close to the real axis, would result in similar behavior. Generally, after the collision, placing the poles closer to the origin resulted in better system behavior. For the plots shown here, the roots of A^* are assigned to 0.75 and 0.2, before and after the collision, respectively.

V. Conclusions

In this paper, a variable-order adaptive controller has been much more effective than similar fixed-order adaptive controllers for control of a flexible system whose order changes abruptly during operation. Interestingly, the best order for the controller after the impact, in the example here, is one smaller than the true order of the plant. These results suggest that the criterion for adaptively adjusting the order should involve the computed control command, as well as a measure of fit to data for the desired model.

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Application of Encke's Method to Long Arc Orbit Determination Solutions

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Introduction

THE Laser Geodynamics Satellite (LAGEOS) was launched to perform geodynamics research in such areas as plate tectonic motion, long wavelength gravity field models, long period diurnal and semidiurnal tides, Earth rotation, and in related areas such as relativity.[1] To accomplish these studies, LAGEOS was constructed as a metal sphere approximately 60 cm in diameter with a mass of 407 kg and launched into a nearly circular orbit with an altitude of about 6000 km. The small area-to-mass ratio and high altitude reduce the effects of nongravitational forces. Embedded in its aluminum skin are 422 cube corner reflectors. LAGEOS is tracked by means of laser range measurements from a global set of ground stations, and to date, over 12 years of laser measurements have been collected. Long arc solutions of LAGEOS' orbit are performed specifically to study long period effects such as the time rates of change of the geopotential coefficients J_2 and J_3, the 18.6-year tidal effects, and other long period effects on the orbit.[2-4]

The numerical integration of the differential equations that model a satellite's orbital motion plays a significant role in the orbit determination process, particularly for precise solutions spanning thousands of orbital revolutions. When the batch filter is used to determine precise orbital solutions, it is affected by the numerical integration process in two ways. First, due to round off and truncation errors, the numerically inte-

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grated solution corresponding to a specific set of initial conditions will diverge from the true solution particularly in the along-track direction. Although these errors can affect the values assigned to the estimated parameters by the batch filter, their effect can be reduced to some extent by using empirical drag-like parameters. Second, for long arc solutions, the numerically integrated solution must properly reflect changes in the orbit due to small changes in the initial conditions predicted by the batch filter. Experience in processing range measurements to LAGEOS for arc lengths on the order of three years of more has shown that the formulation of the differential equations using Cowell's method is not satisfactory when used with the batch filter. Consequently, Encke's method is used to reformulate the differential equations to 1) reduce the effects of numerical integration errors and 2) improve the convergence of the batch filter solution.

Batch Filter

From linear filtering theory, the batch filter assumes the following: 1) a linear dynamical system of the form

$$\dot{x} = A(t)x$$

with the corresponding analytical solution

$$x(t) = \Phi(t,t_0)x(t_0)$$

where Φ is the state transition matrix, and 2) a linear observation-state relationship of the form

$$y(t_i) = H(t_i)x(t_i) + \eta(t_i)$$

where η is the measurement noise. The minimum variance estimate for $x(t_0)$ can be found from

$$(H^T R^{-1} H)x(t_0) = H^T R^{-1} y \tag{1}$$

where

$$H = \begin{bmatrix} H(t_1)\Phi(t_1,t_0) \\ \cdot \\ \cdot \\ \cdot \\ H(t_n)\Phi(t_n,t_0) \end{bmatrix}, \quad y = \begin{bmatrix} y(t_1) \\ \cdot \\ \cdot \\ \cdot \\ y(t_n) \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta(t_1) \\ \cdot \\ \cdot \\ \cdot \\ \eta(t_n) \end{bmatrix}$$

$$R = E\left[\eta\eta^T\right]$$

The application of linear filtering techniques to the problem of determining the initial state vector $X(t_0)$ of a satellite's trajectory, given the nonlinear dynamics of orbital motion and the nonlinear relationship between the range measurements and the state vector, requires the introduction of a known nominal trajectory $X_N(t)$ and the deviation between the true and nominal trajectories $x(t)$ such that

$$x(t) = X(t) - X_N(t)$$

The equations of motion can be written in the form

$$\dot{X} = F(t,X,c)$$

Expanding this equation in a Taylor series about X_N and neglecting higher order terms yields

$$\dot{x} = A(t,X_N)\,x \tag{2}$$

where $A = \partial F/\partial X$. The nonlinear observation-state relationship can be written as

$$Y = g(t,X) + \eta$$

Expanding this equation in a Taylor series about X_N and neglecting higher order terms yields

$$y = Y - Y_N = H(t,X_N)x + \eta \tag{3}$$

where $H = \partial g/\partial X$. Equations (2) and (3) form a linear system to which the batch filter can be applied to determine an estimate of the initial deviation vector $x(t_0)$. The initial value of the true state vector is recovered by determining the initial deviation vector from the normal equations [Eq. (1)] and using the known initial state vector of the nominal trajectory to find

$$X(t_0) = X_N(t_0) + x(t_0)$$

The higher order terms that are neglected in the Taylor series expansions will not be negligible if the differences between the true and nominal trajectories are too large. Consequently, the batch filter algorithm is iterated several times by selecting the latest estimate of the true trajectory to be the nominal trajectory for the next iteration. If the initial selection for the nominal trajectory is sufficiently close to the true trajectory so that the linearizations are valid, the iterative solutions of the batch filter will converge to some "best" trajectory and the magnitude of $x(t_0)$ will tend toward zero. When successive estimates of the true trajectory do not change appreciably, the batch filter has reached convergence.

There are various suitable algorithms that may be used to solve the normal equations [Eq. (1)]. For this work, a linear system solver that makes use of Givens transformations[5] is employed to solve for $x(t_0)$ and to provide a value for the linear predicted root mean square (LPRMS) of the measurement residuals. The LPRMS would be the actual root mean square of the measurement residuals if the observations are linear functions of the state vector. The actual measurement root mean square is compared to the LPRMS for a measure of how well the orbit solution has converged.

Encke's Method

The solution of the batch filter requires the solution of the differential equations that are used to describe the nominal trajectory as well as the state transition matrix along the nominal trajectory. For short arc solutions, this can be accomplished to a reasonable accuracy by numerically integrating the differential equations of the orbital motion in the form (Cowell's method)

$$\ddot{r} = (\mu/r^3)r + f(t,r,\dot{r}) \tag{4}$$

where r and $\dot{r}$ are the position and velocity of the satellite, respectively, μ is the gravitational parameter, and f represents the perturbing forces to the two body motion.

Numerical integration errors will increase in proportion to the length of the trajectory, and while they are unavoidable, various techniques can be used to reduce their effect on the solution. One such technique is Encke's method which involves rewriting the equations of motion that need to be integrated. By defining an Encke difference vector $\epsilon(t)$ as

$$\epsilon(t) = r(t) - r_s(t)$$

where $r_s(t)$ is a reference trajectory with a known analytic solution that satisfies the equation

$$\ddot{r}_s = -(\mu/r_s^3)r_s + f_a(t,r_s,\dot{r}_s) \tag{5}$$

then the differential equation for $\epsilon(t)$ can be written as

$$\ddot{\epsilon} = [-(\mu/r^3)r + (\mu/r_s^3)r_s] + f - f_s \tag{6}$$

The solution to Eq. (4) can be found by numerically integrating Eq. (6) and using the reference trajectory solution to form

$$r(t) = \epsilon(t) + r_s(t)$$

With appropriate attention given to the order of computations and the selection of the reference orbit so that the ratio of the magnitude of ϵ to r remains sufficiently small, then Encke's method will provide a more accurate solution for $r(t)$ than Cowell's method at the expense of a small increase in computational cost.[6,7]

It is also important that the perturbing force f_s be of a form that allows for an analytic solution to Eq. (5) to exist for both the position vector r_s and the velocity vector $\dot{r}_s$. For this analysis, the reference trajectory is modeled using a secularly precessing ellipse[6] where the orbital elements are defined as

$$a = a_0 = \text{const} \qquad e = e_0 = \text{const} \qquad i = i_0 = \text{const}$$

$$\Omega = \Omega_0 + \frac{d\Omega}{df} \times (f - f_0) \qquad \omega = \omega_0 + \frac{d\omega}{df} \times (f - f_0)$$

$$M = M_0 + \dot{M} \times (t - t_0)$$

where the subscript zero indcates values corresponding to $t = t_0$; a, e, and i are the semimajor axis, eccentricity, and inclination, respectively; and Ω, ω, f, and M are the longitude of the ascending node, argument of perigee, true anomaly, and mean anomaly, respectively.

The position along the reference trajectory is expressed as

$$r_s = B(\Omega)C(i)D(u)\begin{bmatrix} r_s \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

where the matrices B, C, and D are defined by

$$B(\Omega) = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C(i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & \sin i & \cos i \end{bmatrix}$$

$$D(u) = \begin{bmatrix} \cos u & \sin u & 0 \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$r_s = [a(1 - e^2)]/(1 + e\cos f)$ and u is the argument of latitude. The velocity $\dot{r}_s$ and acceleration $\ddot{r}_s$ vectors along the reference trajectory are defined as the time derivatives of Eq. (7) (e.g., see Refs. 6–8). The rates for Ω and ω could just as easily be written with respect to time as with respect to the true anomaly. In either case, the rates can be determined analytically using perturbation theory or empirically using a previously determined solution or simulation.

Application of Encke's Method to the Batch Filter

Although Encke's method can reduce the effect of numerical integration errors, its most significant contribution to long arc solutions is the improvement in the iterative application of the batch filter. The sensitivity of the solution at the final point in the trajectory to changes in the initial conditions is proportional to the arc length. As the arc length increases, small changes in the initial conditions can result in large changes in the state during later portions of the trajectory. Consequently, for the batch filter to properly converge, it is critical for the numerically integrated solution to be responsive to small changes in the initial conditions.

If the reference trajectory in the Encke formulation is held fixed during the iterative application of the batch filter, then

$$x(t_0) = X(t_0)_{i+1} - X(t_0)_i$$

$$= [\epsilon(t_0)_{i+1} + r_s(t_0)] - [\epsilon(t_0)_i - r_a(t_0)]$$

$$= \epsilon(t_0)_{i+1} - \epsilon(t_0)_i$$

and the estimate of the correction to $X(t_0)$ is also the correction to $\epsilon(t_0)$. Since ϵ is much smaller in magnitude than X, x will be a larger percentage of ϵ than of X. Thus, the numerically integrated solution for the true trajectory will show a more appropriate response to the correction suggested by the deviation vector when Encke's method is used rather than Cowell's method.

Results

Both Cowell's method and Encke's method were applied to the solution of a six-yera LAGEOS trajectory (approximately 15,000 orbital revolutions). The initial epoch of the solution was taken to be 0^h May 7, 1976 (MJD 42905.0) and involved approximately 100,000 laser range observations from 45 stations. The numerical integration was identical for both applications including integration stepsize, force model, etc. The force model used in this analysis includes a geopotential complete to degree and order 20, solar and Earth radiation pressure, and n-body gravitational effects as well as other contributing forces. The Encke reference trajectory was a secularly precessing ellipse that used mean elements based on the examination of the orbital elements of a 10.6-year orbit solution. These mean orbital elements and rates are given in Table 1. It should be noted that this reference orbit has been used successfully for arc lengths of up to at least 12.8 years.

The actual measurement rms and the LPRMS for the batch filter using Cowell's and Encke's methods are listed in Tables 2 and 3, respectively. The estimated parameters include $x(t_0)$ and a series of C_t parameters, which are constant, empirical

Table 1 Reference orbit mean elements[a]

$a_0 = 12{,}271.0$	$\Omega_0 = 0.516045$ rad	$\frac{d\Omega}{df} = 1.4899198 \times 10^{-4}$
$e_0 = 0.0044$	$\omega_0 = 4.5684342$ rad	$\frac{d\omega}{df} = -9.322410 \times 10^{-5}$
$i_0 = 1.9170697$ rad	$M_0 = 5.6168884$ rad	$\dot{M}_0 = 4.644496 \times 10^{-4}$ rad/s

[a] $t_0 = 0^h$ May 7, 1976.

Table 2 Batch filter results using Cowell's method

Iteration	Measurement rms, m	LPRMS, m
$n+1$	0.519	0.308
$n+2$	2.004	0.308
$n+3$	0.742	0.308
$n+4$	1.197	0.308

Table 3 Batch filter results using Encke's method

Iteration	Measurement rms, m	LPRMS, m
$n+1$	217.172	0.309
$n+2$	22.110	0.309
$n+3$	1.239	0.309
$n+4$	0.312	0.309
$n+5$	0.309	0.309

15-day drag parameters. Tables 2 and 3 list the results after the batch filter had nearly converged using Cowell's method. Notice that the measurement rms in Table 2 exhibits relatively large, seemingly random changes from one iteration to the next indicating that the integrated solution is not properly responding to the updates from the batch filter. Due to differences in the numerical integration errors between the two methods, the initial results in Table 3 appear to degrade. This is a result of changes in the estimated values for the C_t parameters. Once the estimates for C_t converge, the measurement rms rapidly converges to the LPRMS. This is in sharp contrast to the results using Cowell's method, which required several more iterations than those listed in Table 2 before the measurement rms happened to agree with the LPRMS.

Encke's method provides a solution to the convergence problem at a relatively small cost in computer storage and time. Results obtained thus far indicate that Encke's method requires only a 4–6% increase in computer time over that required using Cowell's method. In contrast, double precision arithmetic would alleviate the convergence problem as well but would require a significant increase in computer time and inhibit the vectorization capabilities that are available on many super computers.

Conclusions

Encke's method can be used with the batch filter to improve the convergence characteristics as well as reduce numerical integration errors. This can be accomplished for only a small increase (4–6%) in computer time. Similar results in the batch filter convergence have been exhibited in arc lengths of up to 12.8 years (approximately 31,000 orbital revolutions) and involving a large number of estimated parameters. This approach has become the standard solution technique for long arc solutions of LAGEOS, and a modified approach is being examined for use with Starlette trajectories of 3–5 years in length, which represents 15,600–26,000 orbital revolutions.

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Experimental Modal Analysis for Dynamic Models of Spacecraft

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Introduction

LARGE flexible appendages, such as solar arrays on spacecraft, may cause structural dynamics and control coupling problems; therefore, test-verified analytical dynamic models are required before they are placed into orbit. As space structures become larger and more flexible, ground vibration tests of the complete structure become more difficult, if not impossible. One alternative approach is the experimental component mode synthesis. The component mode synthesis technique[1] is well matured in the field of purely computational finite element analysis, but several obstacles must be overcome if experimentally determined substructural modal data are to be used to provide the modal information for syntheses. These problems are 1) selection of a synthesis algorithm, 2) assurance of the accuracy of the modal data of substructures, 3) measurement of the rotational displacement, and 4) evaluation of the flexibility of the connecting parts. Although scalar elements (spring, mass, and dashpot) have been successfully used in syntheses based on test results, few papers have been presented that deal with the combination of plate/beam substructures.[2,3] To make this a practical technique, experience must be accumulated for each type of structure.

In this Note, an unconstrained component mode synthesis technique based on measured modal data is presented. The examples are solar-array-type structures that are divided into three substructures and four flexible joints (hinges). The difficulty encountered in the synthesis of such a plate-like structure is that the rotational displacements cannot be measured in the usual modal test. This difficulty is overcome by introducing a polynomial approximation for the measured modes. The results synthesized are in good agreement with the test results obtained from the combined structures.

Formulation

In the component mode synthesis method based on measured modal data, the unconstrained mode method is more practical because it is impossible to create an exact fixed-boundary condition for large substructures. The basic equation of the synthesis is written as

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} + [\tilde{M}_c]\{\ddot{q}\} + [\tilde{C}_c]\{\dot{q}\} + [\tilde{K}_c]\{q\} = \{\emptyset\} \tag{1}$$

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